Contents 10

Complex Numbers

Learning outcomes

In this Workbook you will learn what a complex number is and how to combine complex numbers together using the familiar operations of addition, subtraction, multiplication and division. You will also learn how to describe a complex number graphically using the Argand diagram. The connection between the exponential function and the trigonometric functions is explained. You will understand how De Moivre's theorem is used to obtain fractional powers of complex numbers.

Complex Arithmetic 10.1

Introduction

Complex numbers are used in many areas of engineering and science. In this Section we define what a complex number is and explore how two such numbers may be combined together by adding, subtracting, multiplying and dividing. We also show how to find 'complex roots' of polynomial equations.

A **complex number** is a generalisation of an ordinary real number. In fact, as we shall see, a complex number is a pair of real numbers ordered in a particular way. Fundamental to the study of complex numbers is the symbol i with the strange looking property $\mathrm{i}^2 = -1$. Apart from this property complex numbers follow the usual rules of number algebra.

Prerequisites

Before starting this Section you should ...

- be able to add, subtract, multiply and divide real numbers
- be able to combine algebraic fractions together
- understand what a polynomial is
- have a knowledge of trigonometric identities

Learning Outcomes

On completion you should be able to ...

- combine complex numbers together
- find the modulus and conjugate of a complex number
- obtain complex solutions to polynomial equations

1. What is a complex number?

We assume that you are familiar with the properties of ordinary numbers; examples are

$$1,\ -2,\ \frac{3}{10},\ 2.634,\ -3.111,\ \pi,\ e,\ \sqrt{2}$$

We all know how to add, subtract, multiply and divide such numbers. We are aware that the numbers can be positive or negative or zero and also aware of their geometrical interpretation as being represented by points on a 'real' axis known as a number line (Figure 1).

Figure 1

The real axis is a line with a direction (usually chosen to be from left to right) indicated by an arrow. We shall refer to this as the x-axis. On this axis we select a point, arbitrarily, and refer to this as the **origin** O. The origin (where zero is located) distinguishes positive numbers from negative numbers:

- to the right of the origin are the positive numbers
- to the left of the origin are the negative numbers

Thus we can 'locate' the numbers in our example. See Figure 2.

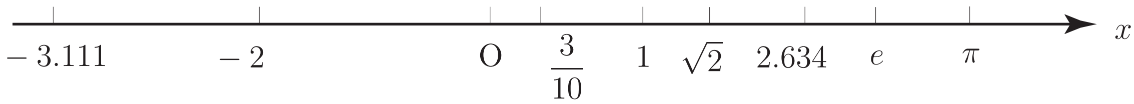


Figure 2

From now on we shall refer to these 'ordinary' numbers as **real** numbers. We can formalise the algebra of real numbers into a set of rules which they obey.
So if x_1, x_2 and x_3 are any three real numbers then we know that, in particular:

1. $x_1 + x_2 = x_2 + x_1 \qquad x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$
2. $1 \times x_1 = x_1 \qquad 0 \times x_1 = 0$
3. $x_1 \times x_2 = x_2 \times x_1 \qquad x_1 \times (x_2 + x_3) = x_1 \times x_2 + x_1 \times x_3$

Also, in multiplication we are familiar with the elementary rules:

(positive) $\times$ (positive) $=$ positive (positive) $\times$ (negative) $=$ negative

(negative) $\times$ (positive) $=$ negative (negative) $\times$ (negative) $=$ positive

It follows that if x represents *any* real number then

$$x^2 \geq 0$$

in words, **the square of a real number is always non-negative**.

In this Workbook we will consider a kind of number (a generalisation of a real number) whose square is not necessarily positive (and not necessarily real either). Don't worry that i 'does not exist'. Because of that it is called imaginary! We just define it and get on and use it and it then turns out to be very useful and important in many practical applications. However, it is important to get to know how to handle complex numbers before using them in calculations. This will not be difficult as the new set of rules is, in fact, precisely the same set of rules obeyed by the 'real' numbers. These new numbers are called complex numbers.

A **complex number** is an ordered pair of real numbers, usually denoted by z or w etc. So if a, b are real numbers then we designate a complex number through:

$$z = a + \mathrm{i}b$$

where i is a symbol obeying the rule

$$\mathrm{i}^2 = -1$$

For simplicity we shall assume we can write

$$\mathrm{i} = \sqrt{-1}.$$

(Often, particularly in engineering applications, the symbol j is used instead of i). Also note that, conventially, examples of actual complex numbers such as $2 + 3\mathrm{i}$ are written like this and not $2 + \mathrm{i}3$. Again we ask the reader to accept matters at this stage without worrying about the meaning of finding the square root of a negative number. Using this notation we can write

$$\sqrt{-4} = \sqrt{(4)(-1)} = \sqrt{4}\sqrt{-1} = 2\mathrm{i} \text{ etc.}$$

Key Point 1

The symbol i is such that

$$\mathrm{i}^2 = -1$$

Using the normal rules of algebra it follows that

$$\mathrm{i}^3 = \mathrm{i}^2 \times \mathrm{i} = -\mathrm{i} \qquad \mathrm{i}^4 = \mathrm{i}^2 \times \mathrm{i}^2 = (-1) \times (-1) = 1$$

and so on.

Simple examples of complex numbers are

$$z_1 = 3 + 2\mathrm{i} \qquad z_2 = -3 + (2.461)\mathrm{i} \qquad z_3 = 17\mathrm{i} \qquad z_4 = 3 + 0\mathrm{i} = 3$$

Generally, if $z = a + \mathrm{i}b$ then 'a' is called the **real part** of z, or $\mathrm{Re}(z)$ for short, and 'b' is called the **imaginary part** of z or $\mathrm{Im}(z)$. The fourth example indicates that the real numbers can be considered a subset of the complex numbers.

Key Point 2

If $\quad z = a + \mathrm{i}b \quad$ then $\quad \mathrm{Re}(z) = a \quad$ and $\quad \mathrm{Im}(z) = b$

Both the real and imaginary parts of a complex number are **real**.

Key Point 3

Two complex numbers $z = a + ib$ and $w = c + id$ are said to be **equal** if and only if both their real parts are the same and both their imaginary parts are the same, that is

$$a = c \qquad \text{and} \qquad b = d$$

Key Point 4

The **modulus** of a complex number $z = a + ib$ is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2}$$

so that the modulus is always a non-negative real number.

Example 1

If $z = 3 - 2\text{i}$ then find $\text{Re}(z)$, $\text{Im}(z)$ and $|z|$.

Solution

Here $\text{Re}(z) = 3$, $\text{Im}(z) = -2$ and $|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$.

Complex conjugate

If $z = a + ib$ is any complex number then the complex conjugate of z is denoted by z^* and is defined by $z^* = a - ib$. (Sometimes the notation $\bar{z}$ is used instead of z^* to denote the conjugate). For example if $z = 2 - 3\text{i}$ then $z^* = 2 + 3\text{i}$. If z is entirely real then $z^* = z$ whereas if z is entirely imaginary then $z^* = -z$. E.g. if $z = 17\text{i}$ then $z^* = -17\text{i}$. In fact the following relationships are easily obtained:

$$\text{Re}(z) = \frac{z + z^*}{2} \qquad \text{and} \qquad \text{Im}(z) = \frac{\text{i}(z^* - z)}{2}$$

If $z = -2 + \mathrm{i}$ find expressions for $\mathrm{Re}(z^*)$ and $\mathrm{Im}(\mathrm{i}(z^* - z))$.

Hint: first find z^*, $z^* - z$, and $\mathrm{i}(z^* - z)$:

Your solution

Answer

$\mathrm{Re}(z^*) = -2$ and $\mathrm{Im}(\mathrm{i}(z^* - z)) = 0$

2. The algebra of complex numbers

Complex numbers are added, subtracted, multiplied and divided in much the same way as these operations are carried out for real numbers.

Addition and subtraction of complex numbers

Let z and w be any two complex numbers

$$z = a + \mathrm{i}b \qquad w = c + \mathrm{i}d$$

then

$$z + w = (a + c) + \mathrm{i}(b + d) \qquad z - w = (a - c) + \mathrm{i}(b - d)$$

For example if $z = 2 - 3\mathrm{i}$, $w = -4 + 2\mathrm{i}$ then

$$z + w = \{2 + (-4)\} + \{(-3) + 2\}\mathrm{i} = -2 - \mathrm{i} \qquad z - w = \{2 - (-4)\} + \{(-3) - 2\}\mathrm{i} = 6 - 5\mathrm{i}$$

Multiplying one complex number by another

In multiplication we proceed using an obvious approach: again consider any two complex numbers $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$. Then

$$\begin{aligned} zw &= (a + \mathrm{i}b)(c + \mathrm{i}d) \\ &= ac + a\mathrm{i}d + \mathrm{i}bc + \mathrm{i}^2 bd \end{aligned}$$

obtained in the usual way by multiplying all the terms in one bracket by all the terms in the other bracket. Now we use the fundamental relation $\mathrm{i}^2 = -1$ so that

$$\begin{aligned} zw &= ac + a\mathrm{i}d + \mathrm{i}bc - bd \\ &= ac - bd + \mathrm{i}(ad + bc) \end{aligned}$$

where we have re-grouped terms with the 'i' symbol and terms without the 'i' symbol separately. These are the real and imaginary parts of the product zw respectively. A numerical example will confirm the approach. If $z = 2 - 3\text{i}$ and $w = -4 + 2\text{i}$ then

$$\begin{aligned} zw &= (2-3\text{i})(-4+2\text{i}) \\ &= 2(-4) + 2(2\text{i}) - 3\text{i}(-4) - 3\text{i}(2\text{i}) \\ &= -8 + 4\text{i} + 12\text{i} - 6\text{i}^2 \\ &= -8 + 16\text{i} + 6 \\ &= -2 + 16\text{i} \end{aligned}$$

Task

If $z = -2 + \text{i}$ and $w = 3 + 2\text{i}$ find expressions for

(a) $z + 2w$, (b) $|z - w|$ and (c) zw

Your solution

(a)

Answer

$z + 2w = 4 + 5\text{i}$

Your solution

(b) Hint: you should find that $z - w = -5 - \text{i}$

Answer

$|z - w| = \sqrt{(-5)^2 + (-1)^2} = \sqrt{26}$

Your solution

(c)

Answer

$zw = -6 + 3\text{i} - 4\text{i} + 2\text{i}^2 = -8 - \text{i}$

In general the square of a complex number is not necessarily a positive real number; it may not even be real at all. For example if $z = -2 + \text{i}$ then

$$z^2 = (-2+\text{i})^2 = 4 - 4\text{i} + \text{i}^2 = 4 - 4\text{i} - 1 = 3 - 4\text{i}$$

However, the product of a complex number with its conjugate is always a non-negative real number. If $z = a + \text{i}b$ then

$$\begin{aligned} zz^* &= (a + \text{i}b)(a - \text{i}b) \\ &= a^2 - a(\text{i}b) + (\text{i}b)a - \text{i}^2b^2 \\ &= a^2 - \text{i}^2b^2 \\ &= a^2 + b^2 \qquad \text{since } \text{i}^2 = -1 \end{aligned}$$

For example, if $z = 2 + \mathrm{i}$ then

$$zz^* = (2 + \mathrm{i})(2 - \mathrm{i}) = 4 + 1 = 5$$

Task

Show, for any complex number $z = a + \mathrm{i}b$ that $zz^* = |z|^2$.

Your solution

Answer

By definition $|z| = \sqrt{a^2 + b^2}$, so that $|z|^2 = a^2 + b^2$. Now $zz^* = a^2 + b^2$ so that $zz^* = |z|^2$.

Dividing one complex number by another

Here we consider the operation of dividing one complex number $z = a + \mathrm{i}b$ by another, $w = c + \mathrm{i}d$:

$$\frac{z}{w} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d}$$

We wish to simplify the right-hand side into the standard form of a complex number (this is called the **Cartesian form)**:

(Real part) $+$ i (Imaginary part)

or the equivalent:

(Real part) $+$ (Imaginary part) i

To do this we multiply 'top and bottom' by the complex conjugate of the bottom (the denominator), that is, by $c - \mathrm{i}d$ (this is called **rationalising**):

$$\frac{z}{w} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d} = \frac{a + \mathrm{i}b}{c + \mathrm{i}d} \times \frac{c - \mathrm{i}d}{c - \mathrm{i}d}$$

and then carry out the multiplication, top and bottom:

$$\begin{aligned} \frac{z}{w} &= \frac{(ac + bd) + \mathrm{i}(bc - ad)}{c^2 + d^2} \\ &= \left(\frac{ac + bd}{c^2 + d^2}\right) + \mathrm{i}\left(\frac{bc - ad}{c^2 + d^2}\right) \end{aligned}$$

which is now in the required form.

The reason for rationalising is to get a real number in the denominator since a complex number divided by a real number is easy to evaluate.

Example 2

Find $\frac{z}{w}$ if $z = 2 - 3\text{i}$ and $w = 2 + \text{i}$.

Solution

$$\begin{aligned}\frac{z}{w} = \frac{2-3\text{i}}{2+\text{i}} &= \frac{(2-3\text{i})\times(2-\text{i})}{(2+\text{i})\times(2-\text{i})} && \text{rationalising}\\ &= \frac{4-3+\text{i}(-6-2)}{4+1} && \text{multiplying out}\\ &= \frac{1}{5}-\frac{8}{5}\text{i} && \text{dividing through}\end{aligned}$$

Task

If $z = 3 - \text{i}$ and $w = 1 + 3\text{i}$ find $\frac{2z+3w}{2z-3w}$.

Your solution

Answer

$$\begin{aligned}\frac{2z+3w}{2z-3w} = \frac{9+7\text{i}}{3-11\text{i}} &= \frac{(9+7\text{i})(3+11\text{i})}{(3-11\text{i})(3+11\text{i})}\\ &= \frac{27-77+(21+99)\text{i}}{9+121}\\ &= -\frac{50}{130}+\frac{120}{130}\text{i} = -\frac{5}{13}+\frac{12}{13}\text{i}\end{aligned}$$

Exercises

1. If $z = 2 - \text{i}$, $w = 3 + 4\text{i}$ find expressions (in standard Cartesian form) for
   (a) $z - 3w$, (b) zw^* (c) $\left(\dfrac{z}{w}\right)^*$ (d) $\left|\dfrac{z}{w}\right|$
2. Verify the following statements for general complex numbers $z = a + \text{i}b$ and $w = c + \text{i}d$
   (a) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ (b) $(zw)^* = z^*w^*$ (c) $\text{Re}(z) = \dfrac{z + z^*}{2}$ (d) $\text{Im}(z) = \dfrac{\text{i}(z^* - z)}{2}$.
3. Find z such that $zz^* + 3(z - z^*) = 13 + 12\text{i}$

Answers

1. (a) $-7 - 13\text{i}$ (b) $2 - 11\text{i}$ (c) $\dfrac{2}{25} + \dfrac{11}{25}\text{i}$ (d) $\dfrac{\sqrt{5}}{5}$
2. Note that since $z^* - z$ is **imaginary** then $\text{i}(z^* - z)$ is **real**!
3. $z = \pm 3 + 2\text{i}$

3. Solutions of polynomial equations

With the introduction of complex numbers we can now obtain solutions to those polynomial equations which may have real solutions, complex solutions or a combination of real and complex solutions. For example, the simple quadratic equation:

$$x^2 + 16 = 0 \qquad \text{can be rearranged:} \qquad x^2 = -16$$

and then taking square roots:

$$x = \pm\sqrt{-16} = \pm 4\sqrt{-1} = \pm 4\text{i}$$

where we are replacing $\sqrt{-1}$ by the symbol 'i'.
This approach can be extended to the general quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{with roots} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so that for example, if

$$3x^2 + 2x + 2 = 0$$

then solving for x:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(3)(2)}}{2(3)} \\ &= \frac{-2 \pm \sqrt{-20}}{6} = \frac{-2 \pm \text{i}\sqrt{20}}{6} \end{aligned}$$

so, (as $\dfrac{\sqrt{20}}{6} = \dfrac{2\sqrt{5}}{6} = \dfrac{\sqrt{5}}{3}$), the two roots are $-\dfrac{1}{3} + \dfrac{\sqrt{5}}{3}\text{i}$ and $-\dfrac{1}{3} - \dfrac{\sqrt{5}}{3}\text{i}$.
In this example we see that the two solutions (roots) are **complex conjugates** of each other. In fact this will always be the case if the polynomial equation has **real** coefficients: that is, if any complex roots occur they will always occur in complex conjugate pairs.

Key Point 5

Complex roots to polynomial equations having **real** coefficients

always occur in **complex conjugate pairs**

Example 3

Given that $x = 3 - 2\mathrm{i}$ is one root of the cubic equation $x^3 - 7x^2 + 19x - 13 = 0$ find the other two roots.

Solution

Since the coefficients of the equation are real and $3-2\mathrm{i}$ is a root then its complex conjugate $3+2\mathrm{i}$ is also a root which implies that $x-(3-2\mathrm{i})$ and $x-(3+2\mathrm{i})$ are factors of the given cubic expression. Multiplying together these two factors:

$$(x-(3-2\mathrm{i}))(x-(3+2\mathrm{i})) = x^2 - x(3-2\mathrm{i}) - x(3+2\mathrm{i}) + 13 = x^2 - 6x + 13$$

So $x^2 - 6x + 13$ is a quadratic factor of the cubic equation. The remaining factor **must** take the form $(x+a)$ where a is real, since only one more linear factor of the cubic equation is required, and so we write:

$$x^3 - 7x^2 + 19x - 13 = (x^2 - 6x + 13)(x+a)$$

By inspection (consider for example the constant terms), it is clear that $a = -1$ so that the final factor is $(x-1)$, implying that the original cubic equation has a root at $x = 1$.

Exercises

1. Find the roots of the equation $x^2 + 2x + 2 = 0$.
2. If i is one root of the cubic equation $x^3 + 2x^2 + x + 2 = 0$ find the two other roots.
3. Find the complex number z if $2z + z^* + 3\mathrm{i} + 2 = 0$.
4. If $z = \cos\theta + \mathrm{i}\sin\theta$ show that $\dfrac{z}{z^*} = \cos 2\theta + \mathrm{i}\sin 2\theta$.

Answers 1. $x = -1 \pm \mathrm{i}$ 2. $-\mathrm{i},\ -2$ 3. $-\dfrac{2}{3} - 3\mathrm{i}$

Argand Diagrams and the Polar Form

10.2

Introduction

In the first part of this Section we introduce a geometrical interpretation of a complex number. Since a complex number $z = x + \mathrm{i}y$ is defined by two real numbers x and y it is natural to consider a plane in which to place a complex number. We shall see that there is a close connection between complex numbers and two-dimensional vectors.

In the second part of this Section we introduce an alternative form, called the polar form, for representing complex numbers. We shall see that the polar form is particularly advantageous when multiplying and dividing complex numbers.

Prerequisites

Before starting this Section you should ...

- know what a complex number is
- be able to use trigonometric functions $\sin$, $\cos$ and $\tan$
- understand what a polynomial is
- possess a knowledge of vectors

Learning Outcomes

On completion you should be able to ...

- represent complex numbers on an Argand diagram
- obtain the polar form of a complex number
- multiply and divide complex numbers in polar form

1. The argand diagram

In Section 10.1 we met a complex number $z = x + \mathrm{i}y$ in which x, y are real numbers and $\mathrm{i}^2 = -1$. We learned how to combine complex numbers together using the usual operations of addition, subtraction, multiplication and division. In this Section we examine a useful geometrical description of complex numbers.

Since a complex number is specified by two real numbers x, y it is natural to represent a complex number by a vector in a plane. We take the usual Oxy plane in which the 'horizontal' axis is the x-axis and the 'vertical' axis is the y-axis.

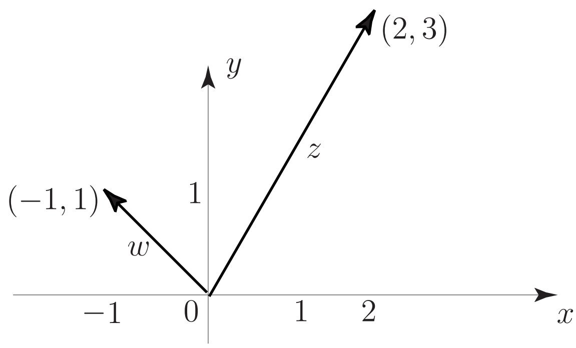


Figure 3

Thus the complex number $z = 2 + 3\mathrm{i}$ would be represented by a line starting from the origin and ending at the point with coordinates $(2, 3)$ and $w = -1 + \mathrm{i}$ is represented by the line starting from the origin and ending at the point with coordinates $(-1, 1)$. See Figure 3. When the Oxy plane is used in this way it is called an **Argand diagram**. With this interpretation the modulus of z, that is $|z|$ is the length of the line which represents z.

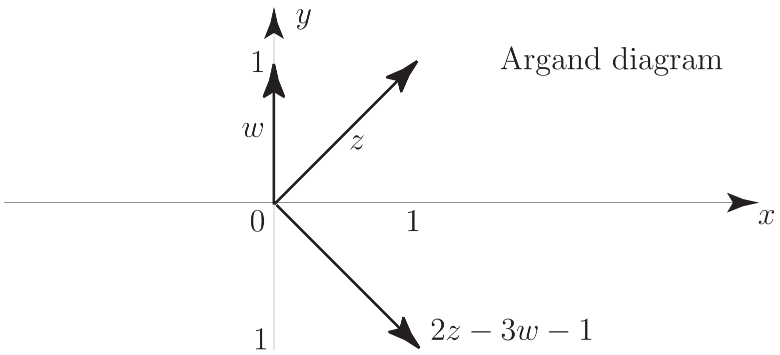
Note: An alternative interpretation is to consider the complex number $a + \mathrm{i}b$ to be represented by the point (a, b) rather than the line from 0 to (a, b).

Task

Given that $z = 1 + \mathrm{i}$, $w = \mathrm{i}$, represent the three complex numbers z, w and $2z - 3w - 1$ on an Argand diagram.

Your solution

Answer
Noting that $2z - 3w - 1 = 2 + 2\mathrm{i} - 3\mathrm{i} - 1 = 1 - \mathrm{i}$ you should obtain the following diagram.



If we have two complex numbers $z = a + \mathrm{i}b$, $w = c + \mathrm{i}d$ then, as we already know

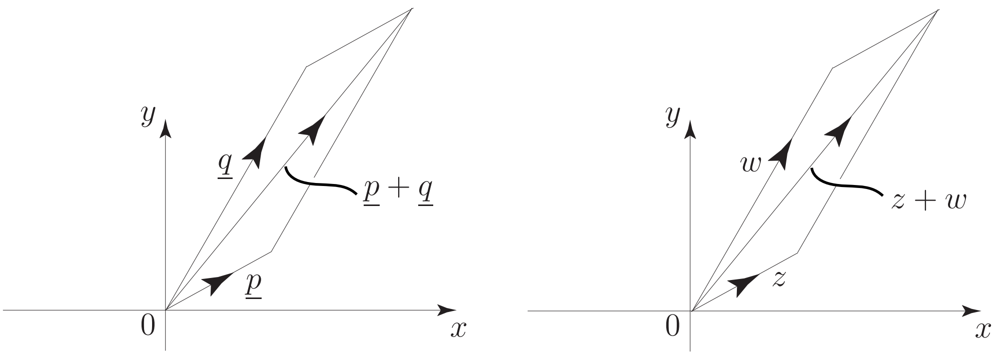
$$z + w = (a + c) + \mathrm{i}(b + d)$$

that is, the real parts add together and the imaginary parts add together. But this is precisely what occurs with the addition of two **vectors**. If $\underline{p}$ and $\underline{q}$ are 2-dimensional vectors then:

$$\underline{p} = a\underline{i} + b\underline{j} \qquad \underline{q} = c\underline{i} + d\underline{j}$$

where $\underline{i}$ and $\underline{j}$ are unit vectors in the x- and y-directions respectively. So, using vector addition:

$$\underline{p} + \underline{q} = (a + c)\underline{i} + (b + d)\underline{j}$$



vector addition complex addition

Figure 4

We conclude from this that addition (and hence subtraction) of complex numbers is essentially equivalent to addition (subtraction) of two-dimensional vectors. (See Figure 4.) Because of this, complex numbers (when represented on an Argand diagram) are **slidable** — as long as you keep their length and direction the same, you can position them anywhere on an Argand diagram.

We see that the Cartesian form of a complex number: $z = a + \mathrm{i}b$ is a particularly suitable form for addition (or subtraction) of complex numbers. However, when we come to consider multiplication and division of complex numbers, the Cartesian description is not the most convenient form that is available to us. A much more convenient form is the polar form which we now introduce.

2. The polar form of a complex number

We have seen, above, that the complex number $z = a + \mathrm{i}b$ can be represented by a line pointing out from the origin and ending at a point with Cartesian coordinates (a, b).

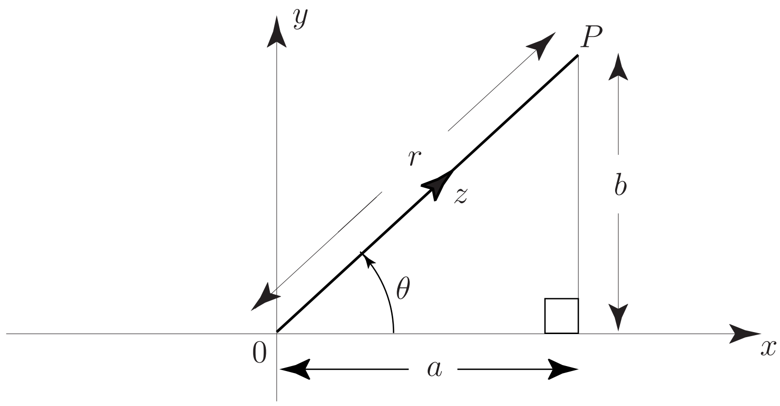


Figure 5

To locate the point P we introduce polar coordinates (r, θ) where r is the positive distance from 0 and θ is the angle measured from the positive x-axis, as shown in Figure 5. From the properties of the right-angled triangle there is an obvious relation between (a, b) and (r, θ):

$$a = r\cos\theta \qquad b = r\sin\theta$$

or equivalently,

$$r = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a}.$$

This leads to an alternative way of writing a complex number:

$$\begin{aligned} z = a + \mathrm{i}b &= r\cos\theta + \mathrm{i}r\sin\theta \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

The angle θ is called the **argument** of z and written, for short, $\arg(z)$. The non-negative real number r is the modulus of z. We normally consider θ measured in radians to lie in the interval $-\pi < \theta \leq \pi$ although any value $\theta + 2k\pi$ for integer k will be equivalent to θ. The angle θ may be expressed in radians or degrees.

Key Point 6

If $z = a + \mathrm{i}b$ then

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

in which

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arg(z) = \tan^{-1}\frac{b}{a}$$

Example 4

Find the polar coordinate form of (a) $z = 3 + 4\text{i}$ (b) $z = -3 - \text{i}$

Solution

(a) Here

$$r = |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \qquad \theta = \text{arg}(z) = \tan^{-1}(\frac{4}{3}) = 53.13^\circ$$

so that $z = 5(\cos 53.13^\circ + \text{i} \sin 53.13^\circ)$

(b) Here

$$r = |z| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10} \approx 3.16 \qquad \theta = \text{arg}(z) = \tan^{-1} \frac{(-1)}{(-3)}$$

It is natural to assume that $\tan^{-1} \dfrac{(-1)}{(-3)} = \tan^{-1} \left(\dfrac{1}{3}\right)$. Using this value on your calculator (unless it is very sophisticated) you will obtain a value of about 18.43° for $\tan^{-1}(\frac{1}{3})$. This is incorrect since if we use the Argand diagram to plot $z = -3 - \text{i}$ we get:

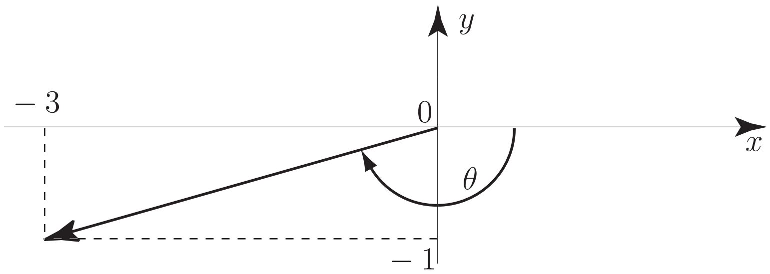


Figure 6

The angle θ is clearly $-180^\circ + 18.43^\circ = -161.57^\circ$.

This example warns us to take care when determining arg(z) purely using algebra. You will always find it helpful to construct the Argand diagram to locate the particular quadrant into which your complex number is pointing. **Your calculator cannot do this for you**.

Finally, in this example, $z = 3.16(\cos 198.43^\circ + \text{i} \sin 198.43^\circ)$.

Task

Find the polar coordinate form of the complex numbers
(a) $z = -\text{i}$ (b) $z = 3 - 4\text{i}$

Your solution

(a)

Answer

$z = 1(\cos 270° + \mathrm{i}\sin 270°)$

Your solution

(b)

Answer

$z = 5(\cos 306.87° + \mathrm{i}\sin 306.87°)$

Remember, to get the correct angle, draw your complex number on an Argand diagram.

Multiplication and division using polar coordinates

The reader will perhaps be wondering why we have bothered to introduce the polar form of a complex number. After all, the calculation of arg(z) is not particularly straightforward. However, as we have said, the polar form of a complex number is a much more convenient vehicle to use for multiplication and division of complex numbers. To see why, let us consider two complex numbers in polar form:

$$z = r(\cos\theta + \mathrm{i}\sin\theta) \qquad w = t(\cos\phi + \mathrm{i}\sin\phi)$$

Then the product zw is calculated in the usual way

$$\begin{aligned} zw &= [r\ (\cos\theta + \mathrm{i}\sin\theta)][t\ (\cos\phi + \mathrm{i}\sin\phi)] \\ &\equiv rt\ [\cos\theta\cos\phi - \sin\theta\sin\phi + \mathrm{i}(\sin\theta\cos\phi + \cos\theta\sin\phi)] \\ &\equiv rt\ [\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)] \end{aligned}$$

in which we have used the standard trigonometric identities

$$\cos(\theta+\phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi \qquad \sin(\theta+\phi) \equiv \sin\theta\cos\phi + \cos\theta\sin\phi.$$

We see that in calculating the product that the moduli r and t *multiply* together whilst the arguments arg(z) $= \theta$ and arg(w) $= \phi$ *add* together.

Task

If $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ find the polar expression for $\dfrac{z}{w}$.

Your solution

Answer

$$\frac{z}{w} = \frac{r}{t}(\cos(\theta-\phi) + \mathrm{i}\sin(\theta-\phi))$$

We see that in calculating the quotient that the moduli r and t **divide** whilst the arguments arg(z) $= \theta$ and arg(w) $= \phi$ **subtract**.

Key Point 7

If $z = r(\cos\theta + \mathsf{i}\sin\theta)$ and $w = t(\cos\phi + \mathsf{i}\sin\phi)$ then

$$zw = rt(\cos(\theta+\phi) + \mathsf{i}\sin(\theta+\phi)) \qquad \frac{z}{w} = \frac{r}{t}(\cos(\theta-\phi) + \mathsf{i}\sin(\theta-\phi))$$

We conclude that addition and subtraction are most easily carried out in Cartesian form whereas multiplication and division are most easily carried out in polar form.

Complex numbers and rotations

We have seen that, when multiplying one complex number by another, the moduli multiply together and the arguments add together. If, in particular, w is a complex number with a modulus t

$$w = t(\cos\phi + \mathsf{i}\sin\phi) \qquad \text{(i.e. } r = t\text{)}$$

and if z is a complex number with modulus 1

$$z = (\cos\theta + \mathsf{i}\sin\theta) \quad \text{(i.e. } r = 1\text{)}$$

then multiplying w by z gives

$$wz = t(\cos(\theta+\phi) + \mathsf{i}\sin(\theta+\phi)) \quad \text{(using Key Point 7)}$$

We see that the effect of multiplying w by z is to *rotate* the line representing the complex number w anti-clockwise through an angle θ which is arg(z), and preserving the length. See Figure 7.

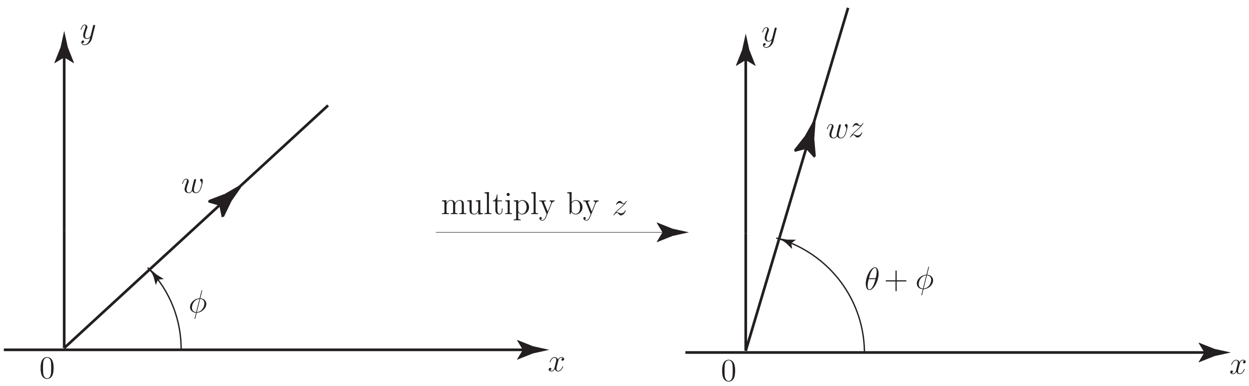


Figure 7

This result would certainly be difficult to obtain had we continued to use the Cartesian form.

Since, in terms of the polar form of a complex number

$$-1 = 1(\cos 180^\circ + \mathsf{i}\sin 180^\circ)$$

we see that multiplying a number by -1 produces a rotation through 180°. In particular multiplying a number by -1 and then by (-1) again (i.e. $(-1)(-1)$) rotates the number through 180° twice, totalling 360°, which is equivalent to leaving the number unchanged. Hence the introduction of complex numbers has 'explained' the accepted (though not obvious) result $(-1)(-1) = +1$.

Exercises

1. Display, on an Argand diagram, the complex numbers $1 - \text{i}$, $1 + 3\text{i}$ and $-1 + 2\text{i}$.

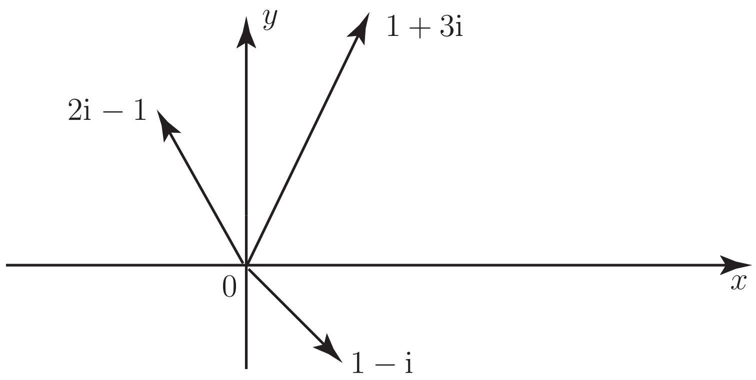
2. Find the polar form of (a) $1 - \text{i}$, (b) $1 + 3\text{i}$ (c) $2\text{i} - 1$. Hence calculate $\dfrac{(1+3\text{i})}{(-1+2\text{i})}$

3. On an Argand diagram draw the complex number $1 + 2\text{i}$. By changing to polar form examine the effect of multiplying $1 + 2\text{i}$ by, in turn, i, i^2, i^3, i^4. Represent these new complex numbers on an Argand diagram.

4. By utilising the Argand diagram convince yourself that $|z+w| \leq |z| + |w|$ for any two complex numbers z, w. This is known as the **triangle inequality**.

Answers

1.



2. (a) $\sqrt{2}(\cos 315^\circ + \text{i} \sin 315^\circ)$ (b) $\sqrt{10}(\cos 71.57^\circ + \text{i} \sin 71.57^\circ)$

   (c) $\sqrt{5}(\cos 116.57^\circ + \text{i} \sin 116.57^\circ)$.

$$\frac{(1+3\text{i})}{(-1+2\text{i})} = \sqrt{2}(\cos(-45^\circ) + \text{i} \sin(-45^\circ)) = \sqrt{2}(\cos(45^\circ) - \text{i} \sin(45^\circ)) = (1 - \text{i}).$$

3. Each time you multiply through by i you effect a rotation through 90° of the line representing the complex number $1 + 2\text{i}$. After four such products you are back to where you started, at $1 + 2\text{i}$.

4. This inequality states that no one side of a triangle is greater in length than the sum of the lengths of the other two sides.

The Exponential Form of a Complex Number 10.3

Introduction

In this Section we introduce a third way of expressing a complex number: the exponential form. We shall discover, through the use of the complex number notation, the intimate connection between the exponential function and the trigonometric functions. We shall also see, using the exponential form, that certain calculations, particularly multiplication and division of complex numbers, are even easier than when expressed in polar form.

The exponential form of a complex number is in widespread use in engineering and science.

Prerequisites

Before starting this Section you should ...

- be able to convert from degrees to radians
- understand how to use the Cartesian and polar forms of a complex number
- be familiar with the hyperbolic functions $\cosh x$ and $\sinh x$

Learning Outcomes

On completion you should be able to ...

- explain the relations between the exponential function e^x and the trigonometric functions $\cos x$, $\sin x$
- interchange between Cartesian, polar and exponential forms of a complex number
- explain the relation between hyperbolic and trigonometric functions

1. Series expansions for exponential and trigonometric functions

We have, so far, considered two ways of representing a complex number:

$z = a + \mathrm{i}b$ Cartesian form

or

$z = r(\cos\theta + \mathrm{i}\sin\theta)$ polar form

In this Section we introduce a third way of denoting a complex number: the **exponential form**.
If x is a real number then, as we shall verify in HELM 16, the exponential number e raised to the power x can be written as a series of powers of x:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

in which $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ is the factorial of the integer n. Although there are an infinite number of terms on the right-hand side, in any practical calculation we could only use a finite number. For example if we choose $x = 1$ (and taking only six terms) then

$$\begin{aligned}\mathrm{e}^1 &\approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}\\ &= 2 + 0.5 + 0.16666 + 0.04166 + 0.00833\\ &= 2.71666\end{aligned}$$

which is fairly close to the accurate value of $\mathrm{e} = 2.71828$ (to 5 d.p.)

We ask you to accept that e^x, for any real value of x, is the **same as** $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ and that if we wish to calculate e^x for a *particular value* of x we will only take a finite number of terms in the series. Obviously the more terms we take in any particular calculation the more accurate will be our calculation.

As we shall also see in HELM 16, similar series expansions exist for the trigonometric functions $\sin x$ and $\cos x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

in which x is measured in radians.

The observant reader will see that these two series for $\sin x$ and $\cos x$ are similar to the series for e^x. Through the use of the symbol i (where $\mathrm{i}^2 = -1$) we will examine this close correspondence.

In the series for e^x replace x on both left-hand and right-hand sides by $\mathrm{i}\theta$ to give:

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + (\mathrm{i}\theta) + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \cdots$$

Then, as usual, replace every occurrence of i^2 by -1 to give

$$\mathrm{e}^{\mathrm{i}\theta} = 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \mathrm{i}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \mathrm{i}\frac{\theta^5}{5!} + \cdots$$

which, when re-organised into real and imaginary terms gives, finally:

$$\begin{aligned} e^{i\theta} &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Key Point 8

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

Example 5

Find complex number expressions, in Cartesian form, for

(a) $e^{i\pi/4}$ (b) e^{-i} (c) $e^{i\pi}$

We use Key Point 8:

Solution

(a) $e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(b) $e^{-i} = \cos(-1) + i\sin(-1) = 0.540 - i(0.841)$ don't forget: use **radians**

(c) $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

2. The exponential form

Since $z = r(\cos\theta + i\sin\theta)$ and since $e^{i\theta} = \cos\theta + i\sin\theta$ we therefore obtain another way in which to denote a complex number: $z = re^{i\theta}$, called the **exponential form**.

Key Point 9

The exponential form of a complex number is

$$z = re^{i\theta} \quad \text{in which} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)$$

so

$$z = re^{i\theta} = r(\cos\theta + i\ \sin\theta)$$

Task

Express $z = 3\mathrm{e}^{\mathrm{i}\pi/6}$ in Cartesian form, correct to 2 d.p.

Use Key Point 9:

Your solution

Answer

$$\begin{aligned} z = 3\mathrm{e}^{\mathrm{i}\pi/6} &= 3(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}) \\ &= 3(0.8660 + \mathrm{i}0.5000) \\ &= 2.60 + 1.50\mathrm{i} \qquad \text{to 2 d.p.} \end{aligned}$$

Example 6

If $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $w = t\mathrm{e}^{\mathrm{i}\phi}$ then find expressions for (a) z^{-1} (b) z^* (c) zw

Solution

(a) If $z = r\mathrm{e}^{\mathrm{i}\theta}$ then $z^{-1} = \dfrac{1}{r\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}$ using the normal rules for indices.

(b) Working in polar form: if $z = r\mathrm{e}^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$ then

$$z^* = r(\cos\theta - \mathrm{i}\sin\theta) = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\mathrm{e}^{-\mathrm{i}\theta}$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. In fact this reflects the general rule: to find the complex conjugate of any expression simply replace i by $-$i wherever it occurs in the expression.

(c) $zw = (r\mathrm{e}^{\mathrm{i}\theta})(t\mathrm{e}^{\mathrm{i}\phi}) = rt\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\phi} = rt\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\phi} = rt\mathrm{e}^{\mathrm{i}(\theta+\phi)}$ which is again the result we are familiar with: when complex numbers are multiplied their moduli multiply and their arguments add.

We see that in some circumstances the exponential form is even more convenient than the polar form since we need not worry about cumbersome trigonometric relations.

Task

Express the following complex numbers in exponential form:

(a) $z = 1 - \mathrm{i}$ (b) $z = 2 + 3\mathrm{i}$ (c) $z = -6$.

Your solution

(a)

Answer

$z = \sqrt{2}\mathrm{e}^{\mathrm{i}7\pi/4}$ (or, equivalently, $\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}$)

Your solution

(b)

Answer

$z = \sqrt{13}\mathrm{e}^{\mathrm{i}(0.9828)}$

Your solution

(c)

Answer

$z = 6\mathrm{e}^{\mathrm{i}\pi}$

3. Hyperbolic and trigonometric functions

We have seen in subsection 1 (Key Point 8) that

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

It follows from this that

$$\mathrm{e}^{-\mathrm{i}\theta} = \cos(-\theta) + \mathrm{i}\sin(-\theta) = \cos\theta - \mathrm{i}\sin\theta$$

Now if we add these two relations together we obtain

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}$$

whereas if we subtract the second from the first we have

$$\sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}$$

These new relations are reminiscent of the hyperbolic functions introduced in HELM 6. There we defined $\cosh x$ and $\sinh x$ in terms of the exponential function:

$$\cosh x = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} \qquad \sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}$$

In fact, if we replace x by $\mathrm{i}\theta$ in these last two equations we obtain

$$\cosh(\mathrm{i}\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2} \equiv \cos\theta \qquad \text{and} \qquad \sinh(\mathrm{i}\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2} \equiv \mathrm{i}\sin\theta$$

Although, by our notation, we have implied that both x and θ are real quantities in fact these expressions for $\cosh$ and $\sinh$ in terms of $\cos$ and $\sin$ are quite general.

Key Point 10

If z is *any* complex number then

$$\cosh(\mathrm{i}z) \equiv \cos z \qquad \text{and} \qquad \sinh(\mathrm{i}z) \equiv \mathrm{i}\sin z$$

Equivalently, replacing z by $\mathrm{i}z$:

$$\cosh z \equiv \cos(\mathrm{i}z) \qquad \text{and} \qquad \mathrm{i}\sinh z \equiv \sin(\mathrm{i}z)$$

Task

Given that $\cos^2 z + \sin^2 z \equiv 1$ for all z then, utilising complex numbers, obtain the equivalent identity for hyperbolic functions.

Your solution

Answer

You should obtain $\cosh^2 z - \sinh^2 z \equiv 1$ since, if we replace z by iz in the given identity then $\cos^2(\mathrm{i}z) + \sin^2(\mathrm{i}z) \equiv 1$. But as noted above $\cos(\mathrm{i}z) \equiv \cosh z$ and $\sin(\mathrm{i}z) \equiv \mathrm{i}\sinh z$ so the result follows.

Further analysis similar to that in the above task leads to **Osborne's rule**:

Key Point 11

Osborne's Rule

Hyperbolic function identities are obtained from trigonometric function identities by replacing $\sin\theta$ by $\sinh\theta$ and $\cos\theta$ by $\cosh\theta$ except that every occurrence of $\sin^2\theta$ is replaced by $-\sinh^2\theta$.

Example 7

Use Osborne's rule to obtain the hyperbolic identity equivalent to $1 + \tan^2\theta \equiv \sec^2\theta$.

Solution

Here $1 + \tan^2\theta \equiv \sec^2\theta$ is equivalent to $1 + \dfrac{\sin^2\theta}{\cos^2\theta} \equiv \dfrac{1}{\cos^2\theta}$. Hence if

$$\sin^2\theta \to -\sinh^2\theta \qquad \text{and} \qquad \cos^2\theta \to \cosh^2\theta$$

then we obtain

$$1 - \frac{\sinh^2\theta}{\cosh^2\theta} \equiv \frac{1}{\cosh^2\theta} \qquad \text{or, equivalently,} \qquad 1 - \tanh^2\theta \equiv \text{sech}^2\theta$$

Engineering Example 1

Feedback applied to an amplifier

Feedback is applied to an amplifier such that

$$A' = \frac{A}{1 - \beta A}$$

where A', A and β are complex quantities. A is the amplifier gain, A' is the gain with feedback and β is the proportion of the output which has been fed back.

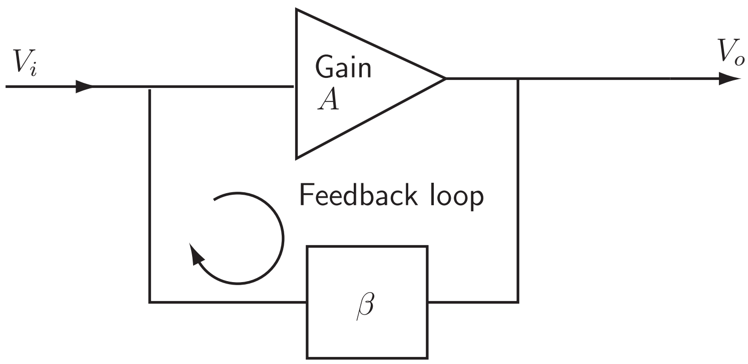


Figure 8: An amplifier with feedback

(a) If at 30 Hz, $A = -500$ and $\beta = 0.005\mathrm{e}^{8\pi \mathrm{i}/9}$, calculate A' in exponential form.

(b) At a particular frequency it is desired to have $A' = 300\mathrm{e}^{5\pi \mathrm{i}/9}$ where it is known that $A = 400\mathrm{e}^{11\pi \mathrm{i}/18}$. Find the value of β necessary to achieve this gain modification.

Mathematical statement of the problem

For (a): substitute $A = -500$ and $\beta = 0.005\text{e}^{8\pi\text{i}/9}$ into $A' = \dfrac{A}{1-\beta A}$ in order to find A'.

For (b): we need to solve for β when $A' = 300\text{e}^{5\pi\text{i}/9}$ and $A = 400\text{e}^{11\pi\text{i}/18}$.

Mathematical analysis

(a) $$A' = \frac{A}{1-\beta A} = \frac{-500}{1-0.005\text{e}^{8\pi\text{i}/9}\times(-500)} = \frac{-500}{1+2.5\text{e}^{8\pi\text{i}/9}}$$

Expressing the bottom line of this expression in Cartesian form this becomes:

$$A' = \frac{-500}{1+2.5\cos(\dfrac{8\pi}{9})+2.5\text{i}\sin(\dfrac{8\pi}{9})} \approx \frac{-500}{-1.349+0.855\text{i}}$$

Expressing both the top and bottom lines in exponential form we get:

$$A' \approx \frac{500\text{e}^{\text{i}\pi}}{1.597\text{e}^{\text{i}2.576}} \approx 313\text{e}^{0.566\text{i}}$$

(b) $$A' = \frac{A}{1-\beta A} \to A'(1-\beta A) = A \to -\beta AA' = A - A'$$

i.e. $$\beta = \frac{A'-A}{AA'} \to \beta = \frac{1}{A} - \frac{1}{A'}$$

So

$$\beta = \frac{1}{A} - \frac{1}{A'} = \frac{1}{400\text{e}^{11\pi\text{i}/18}} - \frac{1}{300\text{e}^{5\pi\text{i}/9}} \approx 0.0025\text{e}^{-11\pi\text{i}18} - 0.00333\text{e}^{-5\pi\text{i}/9}$$

Expressing both complex numbers in Cartesian form gives

$$\beta = 0.0025\cos(-\frac{11\pi}{18}) + 0.0025\text{i}\sin(-\frac{11\pi}{18}) - 0.00333\cos(-\frac{5\pi}{9}) - 0.00333\text{i}\sin(-\frac{5\pi}{9})$$
$$= -2.768\times10^{-4} + 9.3017\times10^{-4}\text{i} = 9.7048\times10^{-4}\text{e}^{1.86\text{i}}$$

So to 3 significant figures $\beta = 9.70\times10^{-4}\text{e}^{1.86\text{i}}$

Exercises

1. Two standard identities in trigonometry are $\sin 2z \equiv 2\sin z \cos z$ and $\cos 2z \equiv \cos^2 z - \sin^2 z$. Use Osborne's rule to obtain the corresponding identities for hyperbolic functions.

2. Express $\sinh(a + \mathrm{i}b)$ in Cartesian form.

3. Express the following complex numbers in Cartesian form (a) $3\mathrm{e}^{\mathrm{i}\pi/3}$ (b) $\mathrm{e}^{-2\pi\mathrm{i}}$ (c) $\mathrm{e}^{\mathrm{i}\pi/2}\mathrm{e}^{\mathrm{i}\pi/4}$.

4. Express the following complex numbers in exponential form

   (a) $z = 2 - \mathrm{i}$ (b) $z = 4 - 3\mathrm{i}$ (c) z^{-1} where $z = 2 - 3\mathrm{i}$.

5. Obtain the real and imaginary parts of $\sinh(1 + \dfrac{\mathrm{i}\pi}{6})$.

Answers

1. $\sinh 2z \equiv 2\sinh z \cosh z$, $\qquad \cosh 2z \equiv \cosh^2 z + \sinh^2 z$.

2. $$\begin{aligned}\sinh(a + \mathrm{i}b) &\equiv \sinh a \cosh \mathrm{i}b + \cosh a \sinh \mathrm{i}b \\ &\equiv \sinh a \cos b + \cosh a(\mathrm{i} \sin b) \\ &\equiv \sinh a \cos b + \mathrm{i} \cosh a \sin b\end{aligned}$$

3. (a) $1.5 + \mathrm{i}(2.598)$ (b) 1 (c) $-0.707 + \mathrm{i}(0.707)$

4. (a) $\sqrt{5}\mathrm{e}^{\mathrm{i}(5.820)}$ (b) $5\mathrm{e}^{\mathrm{i}(5.6397)}$ (c) $2 - 3\mathrm{i} = \sqrt{13}\mathrm{e}^{\mathrm{i}(5.300)}$ therefore $\dfrac{1}{2 - 3\mathrm{i}} = \dfrac{1}{\sqrt{13}}\mathrm{e}^{-\mathrm{i}(5.300)}$

5. $\sinh(1 + \dfrac{\mathrm{i}\pi}{6}) = \dfrac{\sqrt{3}}{2}\sinh 1 + \dfrac{\mathrm{i}}{2}\cosh 1 = 1.0178 + \mathrm{i}(0.7715)$

De Moivre's Theorem 10.4

Introduction

In this Section we introduce De Moivre's theorem and examine some of its consequences. We shall see that one of its uses is in obtaining relationships between trigonometric functions of multiple angles (like $\sin 3x$, $\cos 7x$) and powers of trigonometric functions (like $\sin^2 x$, $\cos^4 x$). Another important use of De Moivre's theorem is in obtaining complex roots of polynomial equations. In this application we re-examine our definition of the argument arg(z) of a complex number.

Prerequisites

Before starting this Section you should ...

- be familiar with the polar form of a complex number
- be familiar with the Argand diagram
- be familiar with the trigonometric identity $\cos^2\theta + \sin^2\theta \equiv 1$
- know how to expand $(x+y)^n$ when n is a positive integer

Learning Outcomes

On completion you should be able to ...

- employ De Moivre's theorem in a number of applications
- fully define the argument arg(z) of a complex number
- obtain complex roots of complex numbers

1. De Moivre's theorem

We have seen, in Section 10.2 Key Point 7, that, in polar form, if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ then the product zw is:

$$zw = rt(\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi))$$

In particular, if $r = 1, t = 1$ and $\theta = \phi$ (i.e. $z = w = \cos\theta + \mathrm{i}\sin\theta$), we obtain

$$(\cos\theta + \mathrm{i}\sin\theta)^2 = \cos 2\theta + \mathrm{i}\sin 2\theta$$

Multiplying each side of the above equation by $\cos\theta + \mathrm{i}\sin\theta$ gives

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = (\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos\theta + \mathrm{i}\sin\theta) = \cos 3\theta + \mathrm{i}\sin 3\theta$$

on adding the arguments of the terms in the product.
Similarly

$$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta.$$

After completing p such products we have:

$$(\cos\theta + \mathrm{i}\sin\theta)^p = \cos p\theta + \mathrm{i}\sin p\theta$$

where p is a positive integer.
In fact this result can be shown to be true for those cases in which p is a negative integer and even when p is a rational number e.g. $p = \frac{1}{2}$.

Key Point 12

If p is a rational number:

$$(\cos\theta + \mathrm{i}\sin\theta)^p \equiv \cos p\theta + \mathrm{i}\sin p\theta$$

This result is known as **De Moivre's theorem**.

Recalling from Key Point 8 that $\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$, De Moivre's theorem is simply a statement of the laws of indices:

$$(\mathrm{e}^{\mathrm{i}\theta})^p = \mathrm{e}^{\mathrm{i}p\theta}$$

2. De Moivre's theorem and root finding

In this subsection we ask if we can obtain fractional powers of complex numbers; for example what are the values of $8^{1/3}$ or $(-24)^{1/4}$ or even $(1+\mathrm{i})^{1/2}$?
More precisely, for these three examples, we are asking for those values of z which satisfy

$$z^3 - 8 = 0 \qquad \text{or} \qquad z^4 + 24 = 0 \qquad \text{or} \qquad z^2 - (1+\mathrm{i}) = 0$$

Each of these problems involve finding roots of a complex number.
To solve problems such as these we shall need to be more careful with our interpretation of arg(z) for a given complex number z.

Arg(z) revisited

By definition arg(z) is the angle made by the line representing z with the positive x-axis. See Figure 9(a). However, as the Figure 9(b) shows you can increase θ by 2π (or 360^0) and still obtain the *same* line in the xy plane. In general, as indicated in Figure 9(c) any integer multiple of 2π can be added to or subtracted from arg(z) without affecting the **Cartesian form** of the complex number.

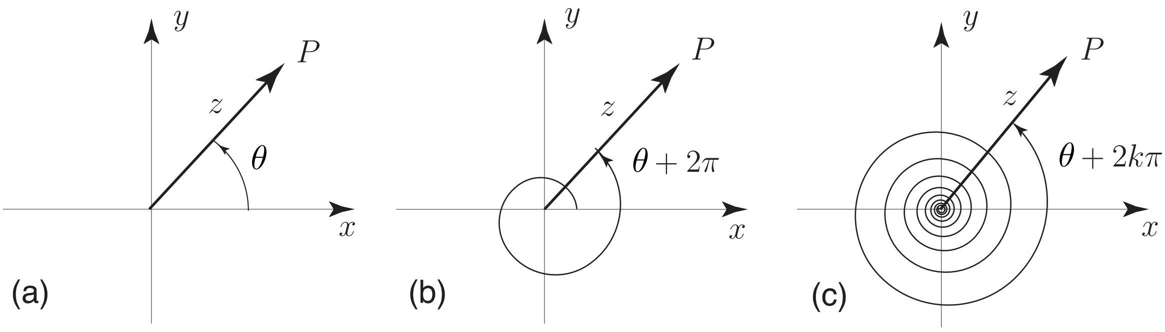


Figure 9

Key Point 13

arg(z) is unique only up to an integer multiple of 2π radians

For example:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}) \qquad \text{in polar form}$$

However, we could also write, equivalently:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2\pi))$$

or, in full generality:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \qquad k = 0, \pm 1, \pm 2, \cdots$$

This last expression shows that in the polar form of a complex number the argument of z, arg(z), can assume infinitely many different values, each one differing by an integer multiple of 2π. This is nothing more than a consequence of the well-known properties of the trigonometric functions:

$$\cos(\theta + 2k\pi) \equiv \cos\theta, \qquad \sin(\theta + 2k\pi) \equiv \sin\theta \qquad \text{for } \textit{any} \text{ integer } k$$

We shall now show how we can use this more general interpretation of arg(z) in the process of finding roots.

Example 8

Find all the values of $8^{1/3}$.

Solution

Solving $z = 8^{1/3}$ for z is equivalent to solving the cubic equation $z^3 - 8 = 0$. We expect that there are three possible values of z satisfying this cubic equation. Thus, rearranging: $z^3 = 8$. Now write the right-hand side as a complex number in polar form:

$$z^3 = 8(\cos 0 + \mathsf{i} \sin 0)$$

(i.e. $r = |8| = 8$ and arg$(8) = 0$). However, if we now generalise our expression for the argument, by adding an arbitrary integer multiple of 2π, we obtain the modified expression:

$$z^3 = 8(\cos(2k\pi) + \mathsf{i} \sin(2k\pi)) \qquad k = 0, \pm 1, \pm 2, \cdots$$

Now take the cube root of both sides:

$$\begin{aligned} z &= \sqrt[3]{8}(\cos(2k\pi) + \mathsf{i} \sin(2k\pi))^{\frac{1}{3}} \\ &= \sqrt[3]{8}(\cos \frac{2k\pi}{3} + \mathsf{i} \sin \frac{2k\pi}{3}) \qquad \text{using De Moivre's theorem.} \end{aligned}$$

Now in this expression k can take any integer value or zero. The normal procedure is to take three consecutive values of k (say $k = 0, 1, 2$). Any other value of k chosen will lead to a root (a value of z) which repeats one of the three already determined.

So if
$$\begin{aligned} k &= 0 \qquad z_0 = 2(\cos 0 + \mathsf{i} \sin 0) = 2 \\ k &= 1 \qquad z_1 = 2(\cos \frac{2\pi}{3} + \mathsf{i} \sin \frac{2\pi}{3}) = -1 + \mathsf{i}\sqrt{3} \\ k &= 2 \qquad z_2 = 2(\cos \frac{4\pi}{3} + \mathsf{i} \sin \frac{4\pi}{3}) = -1 - \mathsf{i}\sqrt{3} \end{aligned}$$

These are the three (complex) values of $8^{\frac{1}{3}}$. The reader should verify, by direct multiplication, that $(-1 + \mathsf{i}\sqrt{3})^3 = 8$ and that $(-1 - \mathsf{i}\sqrt{3})^3 = 8$.

The reader may have noticed within this Example a subtle change in notation. When, for example, we write $8^{1/3}$ then we are expecting **three** possible values, as calculated above. However, when we write $\sqrt[3]{8}$ then we are only expecting **one** value: that delivered by your calculator.

Note the two complex roots are complex conjugates (since $z^3 - 8 = 0$ is a polynomial equation with real coefficients).

In Example 8 we have worked with the polar form. Precisely the same calculation can be carried through using the exponential form of a complex number. We take this opportunity to repeat this calculation but working exclusively in exponential form.
Thus

$$\begin{aligned} z^3 &= 8 \\ &= 8\mathrm{e}^{\mathrm{i}(0)} \qquad (\text{i.e. } r = |8| = 8 \quad \text{and} \quad \arg(8) = 0) \\ &= 8\mathrm{e}^{\mathrm{i}(2k\pi)} \qquad k = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

therefore taking cube roots

$$\begin{aligned} z &= \sqrt[3]{8}\left[\mathrm{e}^{\mathrm{i}(2k\pi)}\right]^{\frac{1}{3}} \\ &= \sqrt[3]{8}\mathrm{e}^{\frac{\mathrm{i}2k\pi}{3}} \qquad \text{using De Moivre's theorem.} \end{aligned}$$

Again k can take any integer value or zero. Any three consecutive values will give the roots.

$$\begin{array}{lrcll} \text{So if} & k & = & 0 & z_0 = 2\mathrm{e}^{\mathrm{i}0} = 2 \\ & k & = & 1 & z_1 = 2\mathrm{e}^{\frac{\mathrm{i}2\pi}{3}} = -1 + \mathrm{i}\sqrt{3} \\ & k & = & 2 & z_2 = 2\mathrm{e}^{\frac{\mathrm{i}4\pi}{3}} = -1 - \mathrm{i}\sqrt{3} \end{array}$$

These are the three (complex) values of $8^{\frac{1}{3}}$ obtained using the exponential form. Of course at the end of the calculation we have converted back to standard Cartesian form.

Task

Following the procedure outlined in Example 8 obtain the two complex values of $(1 + \mathrm{i})^{1/2}$.

Begin by obtaining the polar form (using the general form of the argument) of $(1 + \mathrm{i})$:

Your solution

Answer

You should obtain $1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \quad k = 0, \pm 1, \pm 2, \cdots$.

Now take the square root and use De Moivre's theorem to complete the solution:

Your solution

Answer

You should obtain

$$z_1 = \sqrt[4]{2}(\cos\frac{\pi}{8} + \mathsf{i}\sin\frac{\pi}{8}) = 1.099 + 0.455\mathsf{i}$$
$$z_2 = \sqrt[4]{2}(\cos(\frac{\pi}{8} + \pi) + \mathsf{i}\sin(\frac{\pi}{8} + \pi)) = -1.099 - 0.455\mathsf{i}$$

A good exercise would be to repeat the calculation using the exponential form.

Exercise

Find all those values of z which satisfy $z^4 + 1 = 0$. Write your values in standard Cartesian form.

Answer

$$z_0 = \frac{1}{\sqrt{2}} + \frac{\mathsf{i}}{\sqrt{2}} \qquad z_1 = -\frac{1}{\sqrt{2}} + \frac{\mathsf{i}}{\sqrt{2}} \qquad z_2 = -\frac{1}{\sqrt{2}} - \frac{\mathsf{i}}{\sqrt{2}} \qquad z_3 = \frac{1}{\sqrt{2}} - \frac{\mathsf{i}}{\sqrt{2}}$$